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RM-3808-ARPA
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THE WAKE OF A LARGE BODY
MOVING IN THE IONOSPHERE

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PREFACE

This Memorandum is one result of an investigation of the radar returns from missile wakes. Related Memoranda are RM-3573-ARPA, <u>The Scattering of Electromagnetic Waves from Plasma Cylinders: Part I</u>, and RM-3574-ARPA, <u>The Scattering of Electromagnetic Waves from Plasma Cylinders: Part II</u>.

The work concludes RAND's study of midcourse phenomenology, conducted under the Advanced Research Projects Agency's Defender program.

SUMMARY

This Memorandum investigates the steady-state perturbations of
the ion and electron densities caused by a large-radius conducting
disk moving through the ionosphere at a speed intermediate to the ion
and electron thermal speeds. An approximation to the electron density
in the region close behind the body is found by solving Poisson's
equation, neglecting the ion density. For larger distances behind the
body where the ion and electron densities are almost equal, an ex-
pression for the Fourier transform of the perturbed ion density is
obtained, taking account of the ion thermal motion and scattering in
the electric and magnetic fields, but neglecting collisions. The
transform is inverted to obtain an approximate expression for the
space dependence of the perturbation at large distances. Finally,
the solutions for different body shapes and boundary conditions are
discussed.

ACKNOWLEDGMENT

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SYMBOLS

A = area of the body surface normal to $\vec{V}_B$

$\vec{B}$ = external magnetic field

e = electronic charge

$\vec{e}_o$ = unit vector in direction of $\vec{B}$

f_I = ion-distribution function

g_1 and g_2 = functions of ρ and ζ defined by Eq. (27)

$K_{\parallel}$ = $\vec{K} \cdot \vec{e}_o$

$\vec{K}$ = wave-propagation vector

K_Z = Z-component of $\vec{K}$

$K_{\perp}^2$ = $K^2 - K_{\parallel}^2$

ℓ = integer

m_I = ion mass

N = integral defined in Eq. (45)

n_e = electron density

n_I = ion density

n_K = Fourier transform of perturbed ion density

n_υ = density of ions treated as neutral particles

n_∞ = ambient density of ions and electrons

R_B = body radius

R_L = Larmor radius of the ions

S = source term in Boltzmann equation

S_K = Fourier transform of $S/(V_Z - V_B)f_\alpha$

T_e = electron temperature

T_I = ion temperature

$-u$ = potential far behind the body for the case where the temperature and field scattering of the ions are neglected

u_o = u for $\rho = 0$

u_1 = u for $\rho = 1$

V_B = body speed

V_e = electron thermal speed

V_I = ion thermal speed

w = defined by Eqs. (9), (10), and (11)

Z = z/R_B

$\tilde{Z}$ = $-Z(V_I/V_B)$

$z^{..}$ = defined by Eq. (32)

Z-axis = axis through the center of the body and parallel to $\vec{V}_B$

$Z_G(\rho)$ = surface upon which the zero-ion-temperature solution for the electron density is equal to the ion density

α = $\rho_B^2/8e^{-u_o}$

ζ = $-Z_o^{(u_o-u)/2}$

$\zeta^*(\rho)$ = surface upon which w and $dw/d\zeta$ vanish

$\varkappa$ = $\sqrt{K_x^2 + K_y^2}$

λ^2 = $\left(K^2\rho_L^2/2\right)\left(V_I^2/V_B^2\right)$

λ_D = electron Debye length

μ^2 = $\left(K^2/2K_Z^2\right) V_I^2/V_B^2$

ρ = perpendicular distance from Z-axis divided by R_B

$\tilde{\rho}$ = $(\rho - 1)\rho_B$

ρ_B = R_B/λ_D

ρ_L = R_L/R_B

ϕ = $\dfrac{e\varphi}{kT_e}$

ϕ_K = Fourier transform of ϕ

φ = electrostatic potential

φ_B = electrostatic potential of the body

ψ = angle between $\vec{e}_o$ and $\vec{V}_B$

I. INTRODUCTION

Let us consider the problem of the steady-state perturbations in the medium caused by a conducting disk moving parallel to its axis through the ionsophere under the following conditions:

1. The speed of the body, V_B, is about 10^6 cm/sec, intermediate to the ion thermal speed ($V_I \sim 10^5$ cm/sec) and the electron thermal speed ($V_e \sim 10^7$ cm/sec).

2. The radius of the body, R_B, is large compared to the electron Debye length ($\lambda_D \sim 1$ cm).

3. All ions which strike the body disappear as ions.

4. The Larmor radius of the ions in the earth's magnetic field, R_L, is much larger than the body radius (based on the speed V_B, R_L is of the order of 50 m).

5. The ion collision frequency in the medium is small compared to the ion cyclotron frequency; this condition holds for altitudes above about 130 km.

6. The potential of the body, φ_B, is not much larger in magnitude than kT_e (T_e is the electron temperature).

Under these conditions, the medium can be expected to be quali-tatively that as described by Gurevich.[1] The electrons can be re-garded as being in equilibrium in the potential; because of the great speed of the electrons as compared to that of the body, the body appears to be essentially stationary.

For the ions, the situation is much different. The region direct-ly behind the body will be essentially evacuated of ions. As one moves farther back, the space will gradually fill up with ions because of their random thermal motion and their scattering in the electric field. Still farther back (at distances larger than the ion Larmor radius), the ion density will also be influenced by the magnetic field. It should be pointed out at this time that while the magnetic field is external, the electric field is a consequence of the presence of the moving body. There is a charge on the body itself and a charge den-sity within the medium. The latter results from the different influ-ence of the moving body on the ions and electrons, causing different perturbations in the electron and ion densities.

The problem of estimating the density of ions, taking account of their absorption by the body, their thermal motion, and their scattering in the electric and magnetic fields, would appear at first to be tremendously complicated. However, the situation is somewhat simplified by the fact that close behind the body where the perturbation in the ion density from the ambient value is large, electromagnetic effects on the ion density are minor, and the ion distribution differs little from that of neutral particles having the same mass, temperature, and surface interactions as the ions. Farther back, where the effects of the electric and magnetic fields become important, the perturbation in the ion density from its ambient value is small. Therefore, in the region in which it is necessary to include all influences on the ion density, we can employ small-perturbation theory.

II. GUREVICH ZONES

For the purpose of determining the electrostatic potential,
Gurevich[1] pointed out that the space behind the body can be regarded
as divided into zones.

The zone closest to the body (denoted by Gurevich as the maximum-
rarefaction zone) is one in which the ion density is so low that to a
first approximation it can be neglected in the determination of the
potential.

Farther back (at distances larger than about $R_B(V_B/V_I)$ behind
the body), the ion density determines the potential; except for terms
of order λ_D^2/R_B^2 (assumed here to be small), the electron density equals
the ion density.

The division of the space into these zones can be seen from
Poisson's equation for the potential φ. Using dimensionless coordi-
nates, with the body radius as the unit of length and the origin of
the coordinate system at the center of the disk, Poisson's equation
is given by

$$\frac{1}{\rho_B^2} \nabla^2 \phi = \frac{n_e - n_I}{n_\infty} \tag{1}$$

where

$\phi = e\varphi/kT_e$, a measure of the ratio of the potential to
   the kinetic energy of the electrons

$1/\rho_B^2 = \lambda_D^2/R_B^2 \ll 1$, i.e., the ratio of Debye length to body
   radius is small

n_e, n_I = the electron and ion densities, respectively

n_∞ = the ambient density of both kinds of charged particles

The conditions on ϕ are

$\phi = \phi_B$ on the body $\phi \to 0$ at infinity

and ϕ and its derivatives are continuous (except the normal derivative on the surface of the body).

Since the electrons are essentially in equilibrium in the potential, the electron density is given by

$$\frac{n_e}{n_\infty} = e^\phi \tag{2}$$

The ion density is as yet to be determined. However, we know that there will be a region close behind the body in which the ion density will be small enough so that $n_I/n_\infty < e^\phi$. In this region, as a first approximation, we can neglect n_I/n_∞ altogether, and Poisson's equation becomes

$$\frac{1}{\rho_B^2} \nabla^2 \phi = e^\phi \tag{3}$$

Gurevich pointed out that in the maximum-rarefaction zone, which is the region where Eq. (3) is applicable, the solution e^ϕ will have an order of magnitude of $1/\rho_B^2$. Denoting ϕ within the maximum-rarefaction zone as ϕ_{MRZ}, the zero-order approximation to ϕ_{MRZ} as given by Gurevich is

$$\phi_{MRZ}^{(0)} = \ln \frac{1}{\rho_R^2} \tag{4}$$

Gurevich did not obtain the next approximation in this region.

Outside the maximum-rarefaction zone, where n_I/n_∞ becomes comparable to e^ϕ, Gurevich pointed out that a solution to Poisson's equation can be found by an iteration procedure. This is effected by writing Poisson's equation in the form

$$\phi = \ln \left[\frac{n_I}{n_\infty} + \frac{1}{\rho_B^2} \nabla^2 \phi \right] \tag{5}$$

For the zero-order approximation to ϕ, the ion and electron densities are equated:

$$\phi^{(0)} = \ln \frac{n_I}{n_\infty} \tag{o}$$

The next approximation is

$$\phi^{(1)} = \ln \left(\frac{n_I}{n_\infty} + \frac{1}{\rho_B^2} \nabla^2 \ln \frac{n_I}{n_\infty} \right) \tag{7}$$

and so forth. At large enough distances behind the body, the zero-order approximation (Eq. (6)) suffices. The problem in this region, then, is to determine n_I. Gurevich did this by first treating the ions as neutral particles and then taking account of the magnetic field. He did not consider the effects of the electric field on the ion density.

In this Memorandum an attempt will first be made to find a solution for ϕ_{MRZ} by considering a highly idealized model in which the ions have zero temperature and in which the influence of the electric and magnetic fields on the ion motion is ignored. The solution for this model should approximate the actual solution within the maximum-rarefaction zone.

Second, an attempt will be made to estimate the ion density, taking account of the thermal motion of the ions but not their scattering in the electric and magnetic fields. Since this approximation to the ion density is a fairly good one for distances behind the body less than $(V_B/V_I)R_B$, it can be used to define the "boundary" separating the maximum-rarefaction zone from the rest of the space behind the body. This boundary, which is, of course, fictitious and merely a mathematical convenience, will be defined as the surface upon which the ion density is equal to $e^{\phi_{MRZ}}$.

Finally, the ion density will be considered, taking account of the electric and magnetic fields, in the region where the ion and

electron densities are essentially equal and Eq. (6) is applicable. The effect of collisions will be neglected; it is understood, therefore, that our solution is invalid for distances behind the body larger than the collision mean free path.

III. SOLUTION WHERE TEMPERATURE AND FIELD SCATTERING OF IONS ARE NEGLECTED

If the thermal speed and scattering of the ions in the electric and magnetic fields are neglected, then the ion density will be zero in a semi-infinite cylinder of radius R_B behind the body and equal to the ambient density everywhere else. Using cylindrical coordinates Z, ρ, and θ, Eq. (1) becomes

$$\frac{1}{\rho_B^2}\left[\frac{\partial^2 \phi}{\partial Z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial \phi}{\partial \rho}\right)\right] = e^{\phi} - 1 + \left(\frac{1 - \text{sign}\,(Z)}{2}\right) \qquad \text{for } \rho < 1$$

$$\text{(8)}$$

$$\frac{1}{\rho_B^2}\left[\frac{\partial^2 \phi}{\partial Z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial \phi}{\partial \rho}\right)\right] = e^{\phi} - 1 \qquad \text{for } \rho > 1$$

Negative Z refers to the region behind the body.

The boundary conditions are the same as those for Eq. (1) except for the condition on ϕ as $Z \to -\infty$. An assumption that $\phi \to 0$ as $Z \to -\infty$ implies that the electron density becomes equal to its ambient value as $Z \to -\infty$. However, for V_I identically zero, the ion density is zero in the entire semi-infinite cylindrical region $Z < 0$, $\rho < 1$. A large charge separation such as this is physically unreasonable. We would expect, rather, that the electron density would remain depressed below its ambient value throughout the ion "hole," implying a nonvanishing negative potential behind the body extending to infinity for $\rho < 1$. Indeed, we would expect that far behind the body, ϕ becomes essentially independent of Z. Therefore, the condition that $\phi \to 0$ as $Z \to -\infty$ will be replaced by the physically sensible condition that as $Z \to -\infty$, ϕ becomes a function of ρ only.

Therefore, we will assume that ϕ can be written in the form

$$\phi = -u(\rho)\left(\frac{1 - \text{sign}\,(Z)}{2}\right) + w(\rho,\,Z) \qquad \text{(9)}$$

where $w(\rho, Z) \to 0$ as $|Z| \to \infty$, and where $u(\rho)$ satisfies the following equations and boundary conditions:

$$\frac{1}{\rho_B^2} \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{du}{d\rho} \right) = -e^{-u} \qquad \rho < 1 \qquad (10a)$$

$$\frac{1}{\rho_B^2} \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{du}{d\rho} \right) = 1 - e^{-u} \qquad \rho > 1 \qquad (10b)$$

$$\left(\frac{du}{d\rho} \right)_{\rho=0} = 0$$

$$u \to 0 \qquad \text{as} \qquad \rho \to \infty$$

and u and $du/d\rho$ are continuous at $\rho = 1$. The function w will satisfy

$$\frac{1}{\rho_B^2} \left[\frac{\partial^2 w}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial w}{\partial \rho} \right) \right] = e^w - 1 \qquad Z > 0, \text{ all } \rho \quad (11a)$$

$$\frac{1}{\rho_B^2} \left[\frac{\partial^2 w}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial w}{\partial \rho} \right) \right] = e^{-u}(e^w - 1) \qquad Z < 0, \text{ all } \rho \quad (11b)$$

$$w \to 0 \qquad \text{as} \qquad |Z| \to \infty$$

$$\left(\frac{\partial w}{\partial \rho} \right)_{\rho=0} = 0$$

$$\left. \begin{array}{l} (w - u)_{Z=0-} = \Phi_B \\[2em] w_{Z=0+} = \Phi_B \end{array} \right\} \quad \rho < 1$$

$$(w - u)_{Z=0-} = w_{Z=0+}$$

$$\left.\left(\frac{\partial w}{\partial Z}\right)\right|_{Z=0-} = \left.\left(\frac{\partial w}{\partial Z}\right)\right|_{Z=0+}$$

$$\rho > 1$$

We will first look at the function $u(\rho)$, which determines the electron density far behind the body, and then examine the function w.

SOLUTION FAR BEHIND THE BODY

Interior Solution (Arbitrary ρ_B)

Equation (10a) can be solved analytically:

$$e^{-u} = \frac{8}{\rho_B^2} \frac{\alpha}{\left(1 - \alpha\rho^2\right)^2} \qquad \rho \leq 1 \qquad (12)$$

where

$$\alpha = \frac{\rho_B^2}{8} e^{-u_o}$$

and u_o is the value of u on the axis ($\rho = 0$). The value of u on the axis is as yet unknown; it is determined by matching the inside ($\rho \leq 1$) solution to the outside ($\rho \geq 1$) solution at $\rho = 1$. In terms of α and ρ_B, u and its derivative at $\rho = 1$ are given by

$$1 - \alpha = \frac{-1 + \sqrt{1 + \frac{\rho_B^2}{2} e^{-u_1}}}{\frac{\rho_B^2}{4} e^{-u_1}} \qquad (13a)$$

$$\left(\frac{du}{d\rho}\right)^2_{\rho=1} = 2\rho_B^2 \alpha\epsilon^{-u_1} \qquad (13b)$$

where u_1 is the value of u at $\rho = 1$.

We will use Eqs. (13a) and (13b) later on. One thing we can see immediately from the inside solution is that α must be less than one in order for e^{-u} to be finite for all $\rho \leq 1$. Thus, n_e/n_∞ must be less than $8/\rho_B^2$ on the axis. As would be expected, the electron density near the axis gets very small as the radius of the "hole" gets very large.

Potential at $\rho = 1$

Before attempting to find a solution for $\rho \geq 1$, let us first see how much information can be obtained from Eqs. (10b) and (13) without getting an explicit solution to Eq. (10b). Let us rewrite Eq. (10b) as follows:

$$\frac{1}{\rho_B^2} \frac{d^2 u}{d\rho^2} = 1 - e^{-u} - \frac{1}{\rho_B^2} \frac{1}{\rho} \frac{du}{d\rho} \tag{14}$$

Multiplying both sides of the equation by du and integrating from 0 to u_1, we get

$$\frac{1}{2\rho_B^2} \left(\frac{du}{d\rho}\right)^2_{\rho=1} = e^{-u_1} + u_1 - 1 - \frac{1}{\rho_B^2} \int_o^{u_1} \frac{1}{\rho(u)} \frac{du}{d\rho} du \tag{15}$$

Combining Eqs. (13b) and (15), we get

$$1 - u_1 - e^{-u_1} (1 - \alpha) = - \frac{1}{\rho_B^2} \int_o^{u_1} \frac{1}{\rho(u)} \frac{du}{d\rho} du \tag{16}$$

Since du/dρ is negative, the right-hand side of Eq. (16) is positive. This means that since $\alpha < 1$, u_1 must be less than one for all values of ρ_B. Thus, for large ρ_B, the term $\rho_B^2 e^{-u_1}$ will be large, and Eq. (13a) becomes

$$1 - \alpha \approx \frac{4}{\rho_B} \frac{e^{u_1/2}}{\sqrt{2}} \tag{17}$$

Moreover, since both $(1 - \alpha)$ and the right-hand side of Eq. (16) are of order $1/\rho_B$, u_1 differs from unity by a term of order $1/\rho_B$ for large ρ_B. Thus, we have

$$1 - \alpha \cong \frac{4}{\rho_B}\sqrt{\frac{e}{2}} \qquad \rho_B \gg 1 \tag{18}$$

$$u_1 \sim 1$$

Interior Solution for Large ρ_B

Knowing α, Eq. (12) can now be rewritten for large ρ_B:

$$e^{-u} = \frac{8}{\rho_B^2} \frac{2}{\left[1 - \left(1 - \frac{4}{\rho_B}\sqrt{\frac{e}{2}}\right)\rho^2\right]^2} \tag{19}$$

Since the electron density far behind the body is given by $n_\infty e^{-u}$, it can be seen from Eq. (19) that n_e/n_∞ is very small in the interior of the large-radius ion "hole." On the axis, n_e/n_∞ is equal to $8/\rho_B^2$ and rises fairly slowly with increasing radius up to a distance of a few Debye lengths from the edge of the cylinder. Thereafter, n_e/n_∞ rises rapidly until $\rho = 1$. At $\rho = 1$, it is equal to about $1/e$, provided that ρ_B is large. Thus, a simplified picture of the region $\rho \leq 1$ far behind the large-radius body is one of a cylindrical hole devoid of ions and low in electron density, except for a sheath of thickness of a few Debye lengths at the edge. The electron density in the sheath is given by

$$\frac{n_e}{n_\infty} \cong \frac{2}{\left(|\tilde{\rho}| + \sqrt{2e}\right)^2} \tag{20}$$

where $\tilde{\rho} = (\rho - 1)\rho_B$. Thus, the electron density inside the cylinder near the edge depends upon the distance $|\tilde{\rho}|$ in Debye lengths from the edge of the cylinder, not on the value of ρ_B for large ρ_f.

Solution for $\rho \geq 1$ for Large ρ_B

For large ρ_B, it is convenient to rewrite Eq. (10b) as follows:

$$u''(\tilde{\rho}) + \frac{1}{\rho_B(1 + \tilde{\rho}/\rho_B)} u'(\tilde{\rho}) = 1 - e^{-u} \tag{21}$$

Neglecting terms of order $1/\rho_B$, Eq. (21) becomes

$$u''(\tilde{\rho}) = 1 - e^{-u} \tag{22}$$

One integration gives

$$\frac{1}{2}\left(\frac{du}{d\tilde{\rho}}\right)^2 = e^{-u} + u - 1 \tag{23}$$

so that

$$\frac{1}{\sqrt{2}} \int_u^{u_1} \frac{du}{\sqrt{e^{-u} + u - 1}} = \tilde{\rho} \tag{24}$$

Since $u_i \sim 1$ and u is a monotonically decreasing function of $\tilde{\rho}$, we can expand the exponential e^{-u}. Keeping terms up to u^3, we get

$$u \approx e^{-\tilde{\rho}}\left(1 + \frac{1}{6}(1 - e^{-\tilde{\rho}})\right) \tag{25}$$

Since u decreases exponentially with $\tilde{\rho}$, the electron density becomes essentially equal to its ambient density within a few Debye lengths from the edge of the cylinder. Moreover, we see that the electron density near the edge outside as well as inside the cylindrical hole

depends upon the distance in Debye lengths from the edge and not on the value of ρ_B, as long as ρ_B is large.

The electron density and the absolute value of the charge density are plotted as functions of $\tilde{\rho}$ in Figs. 1 and 2 for $\rho_B = 20$, 50, and 100. Figure 2 shows that the absolute value of the charge density (divided by en_∞) becomes small within a few Debye lengths from the edge of the cylinder, as would be expected.

ELECTRON DENSITY NEAR THE DISK

In order to obtain the potential near the body, we need to estimate the function $w(\rho, Z)$.

Solution Inside the Hole for Large ρ_B

We will find an approximate solution to Eq. (11b) inside the hole for large ρ_B by first rewriting the equation in terms of ρ and a new variable, ζ, where

$$\zeta = -Ze^{(u_o-u)/2} \tag{26}$$

Equation (11b) becomes

$$\frac{1}{8}\frac{\partial^2 w}{\partial\zeta^2} = e^w - g_1(\rho, \zeta) - g_2(\rho, \zeta) \tag{27}$$

where

$$g_1 = \frac{1}{2}\,\zeta\,\frac{\partial w}{\partial\zeta} + \left[1 + \frac{e^{u-u_o}}{8}\left(\frac{\partial^2 w}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial\rho}\right)\right]$$

$$g_2 = \frac{\zeta\rho^2}{16}\left[\frac{\partial w}{\partial\zeta} + \zeta\frac{\partial^2 w}{\partial\zeta^2} + \frac{e^{(u-u_o)/2}}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\partial w}{\partial\zeta}\right)\right]$$
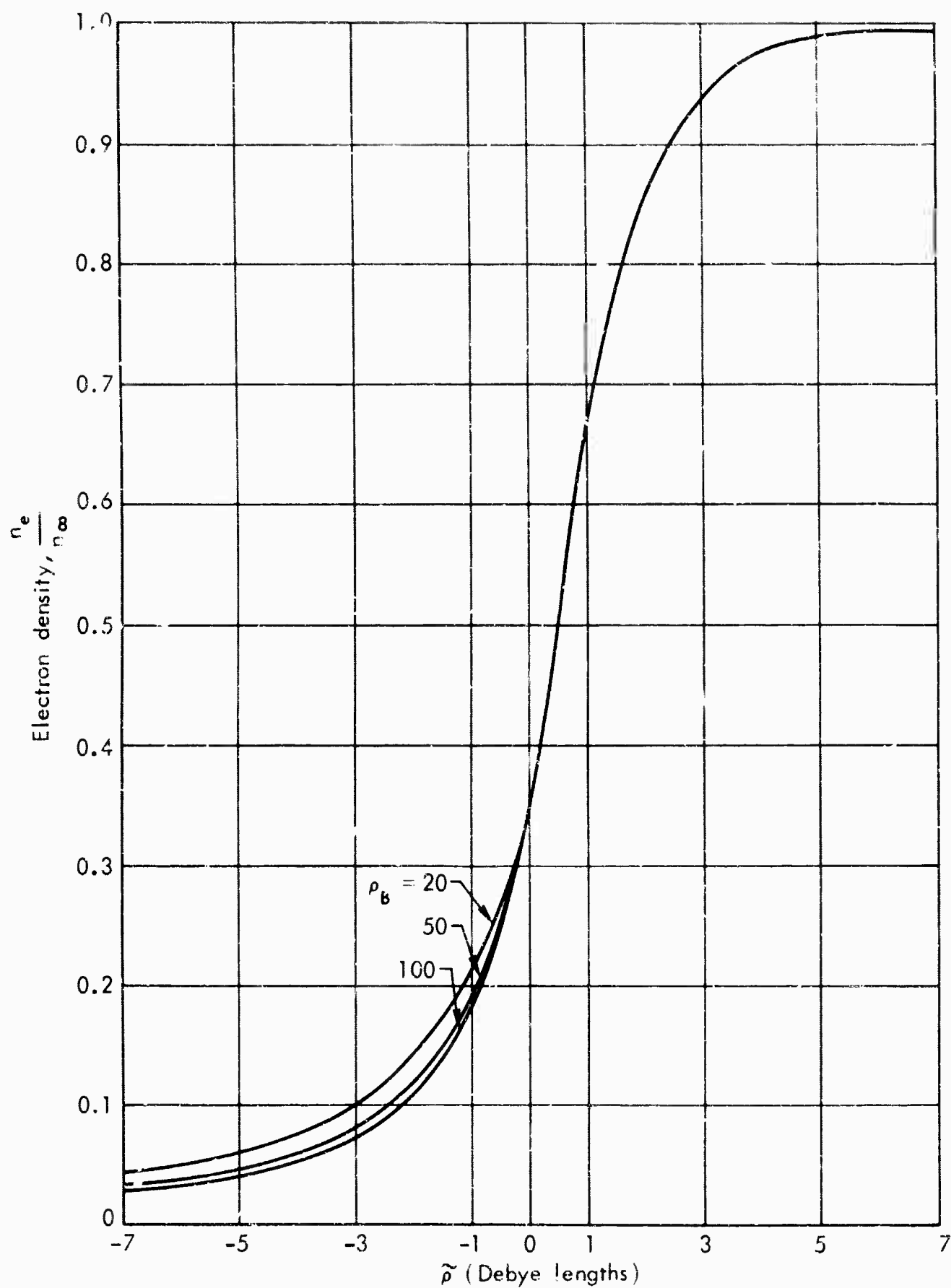
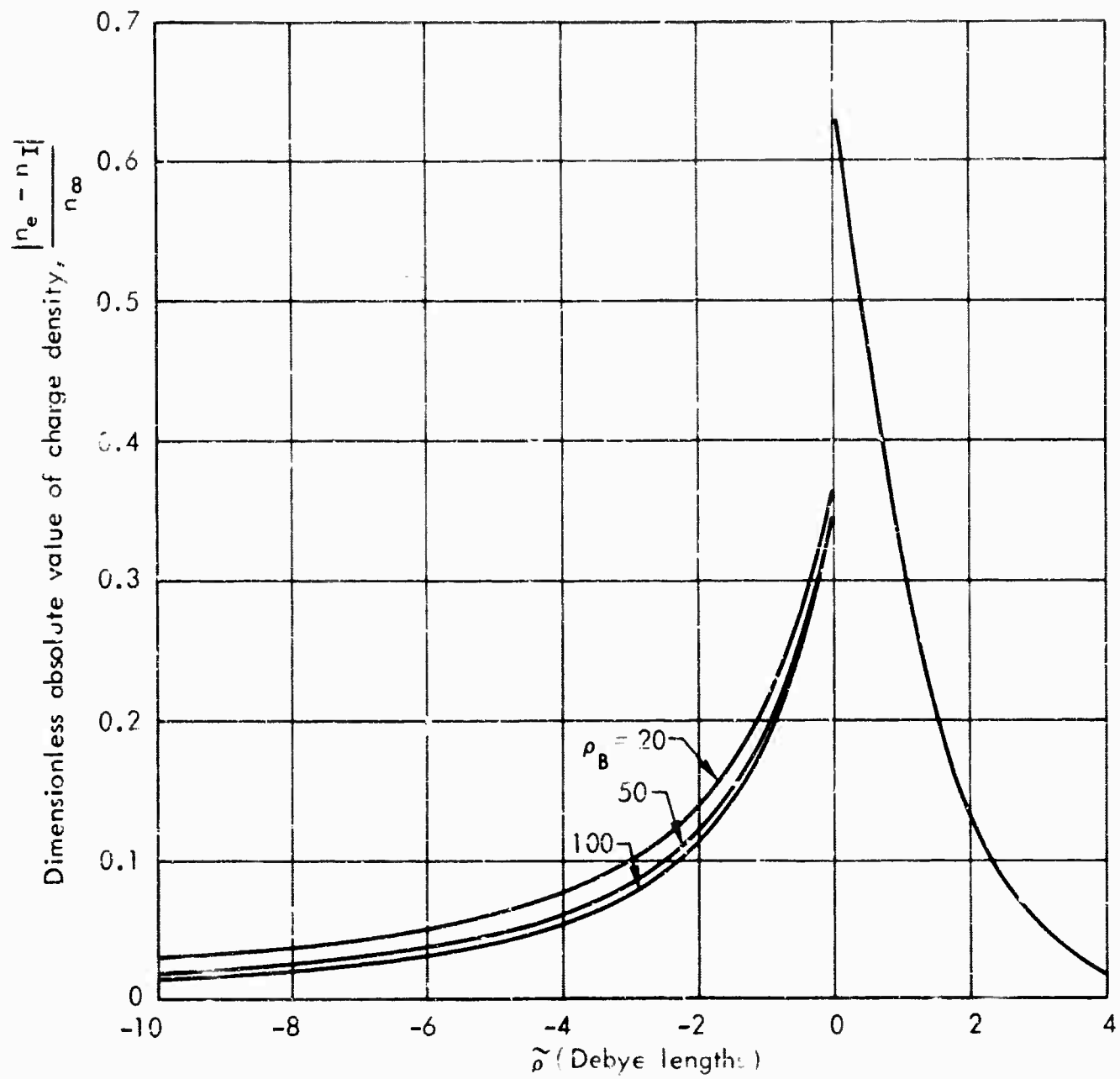Fig.1—Electron density far behind the body for
zero ion temperature

Fig. 2—Absolute value of the charge density far behind
the body for zero ion temperature

The function g_2 vanishes on the axis for all ζ and on the surface of the disk for all $\rho < 1$. For ζ and ρ small but finite, g_2 can be expected to be small in magnitude. The function g_1 will be of order unity for large ζ when w becomes small. For small ζ, $\Phi = \Phi_B$ (independent of ρ), so that

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \, \frac{\partial w}{\partial \rho} \right) \approx \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \, \frac{\partial u}{\partial \rho} \right) = -8 e^{u_o - u}$$

Thus

$$1 + \frac{1}{8} e^{(u - u_o)} \left(\frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial w}{\partial \rho} \right) \to 0 \qquad \text{as } \zeta \to 0$$

Therefore, for small ζ, both g_1 and g_2 will be small in magnitude, becoming identically zero on the surface of the disk. The term e^w, on the other hand, will be large for small ζ and ρ if ρ_B is large and if $-\Phi_B$ is of order unity. If we restrict ourselves to values of $-\Phi_B \leq 1$, then $e^w \geq 1$ for all $\rho \leq 1$. We also know that for large ζ, w will approach zero.

Thus, it seems reasonable to replace Eq. (27) by the following:

$$\frac{1}{8} \frac{d^2 w}{d\zeta^2} = e^w \qquad\qquad \zeta \leq \zeta^*(\rho) \qquad (28)$$

$$w = \Phi_B + u(\rho) \equiv w_B(\rho) \qquad \zeta = 0$$

$$w = \frac{dw}{d\zeta} = 0 \qquad\qquad \zeta \geq \zeta^*(\rho)$$

The function $\zeta^*(\rho)$ is as yet unknown; it is the surface upon which both w and $dw/d\zeta$ vanish and is determined from the solution to Eq. (28).

One integration of Eq. (28) yields

$$\frac{1}{16} \left(\frac{dw}{d\zeta}\right)^2 = e^w - 1 \tag{29}$$

The constant of integration was chosen to cause $dw/d\zeta$ to vanish when $w = 0$. One more integration gives

$$\sqrt{e^w - 1} = \frac{\sqrt{e^{w_B} - 1} - \tan 2\zeta}{1 + \sqrt{e^{w_B} - 1} \, \tan 2\zeta} \tag{30}$$

or

$$e^w = \frac{e^{w_B} \sec^2 2\zeta}{\left(1 + \sqrt{e^{w_B} - 1} \, \tan 2\zeta\right)^2}$$

$$\zeta \le \frac{1}{2} \cos^{-1} e^{-w_B/2} = \zeta^*$$

The function $\zeta^*(\rho)$ is then given by

$$\zeta^*(\rho) = \frac{1}{2} \cos^{-1}\left(e^{(u-\Phi_B)/2}\right) \tag{31}$$

Written in terms of Z rather than ζ, Eq. (31) becomes

$$Z^*(\rho) = -\zeta^* e^{(u-u_o)/2} \tag{32}$$

$$Z^*(\rho) = -\frac{(1 - \alpha\rho^2)}{2} \cos^{-1}\left(\frac{\sqrt{8}}{\rho_B} \frac{e^{-\Phi_B/2}}{(1 - \alpha\rho^2)}\right)$$

with α given by Eq. (18).

The value of $-Z^*$ will vary from approximately $\pi/4$ at $\rho = C$ to a value near zero as $\rho \to 1$. We can see from Eq. (32) that the arc cosine

term will be about $\pi/2$ except for ρ near 1; for ρ near 1, the $1 - \alpha\rho^2$ factor will become very small, since α is near unity. Therefore, an excellent approximation of $Z^*_{approx}(\rho)$ to Eq. (32) for large ρ_B and $-\Phi_B$ of order unity is

$$-Z^*_{approx}(\rho) \cong \frac{\pi}{4} (1 - \rho^2) \tag{33}$$

independent of ρ_B and Φ_B. As an illustration, both Eq. (32) and Eq. (33) are plotted in Fig. 3 for $\rho_B = 100$ and $-\Phi_B = 1$.

We will not attempt to obtain a more accurate solution for e^w than that given by Eq. (30). The error involved in replacing Eq. (27) by Eq. (28) becomes large when w becomes small; however, this error is not likely to be larger than other errors inherent in our approximate theory. Furthermore, it should be noted that the error in our approximations is only significant if we are interested in derivatives of the electron density. The electron density itself is proportional to e^w, and when w is small, errors in w of large percentage will not matter very much.

Thus, we have for the potential behind the disk inside the large-radius cylindrical hole

$$e^{\Phi_{MRZ}} = e^{-u+w} = \frac{e^{\Phi_B} \sec^2 2\zeta}{\left(1 + \sqrt{e^{u+\Phi_B} - 1}\ \tan 2\zeta\right)^2} \qquad \zeta \le \zeta^*$$

$$e^{\Phi_{MRZ}} = e^{-u} \qquad\qquad\qquad\qquad\qquad \zeta \ge \zeta^* \tag{34}$$

In terms of Z and ρ, this gives, for ρ not too near 1,
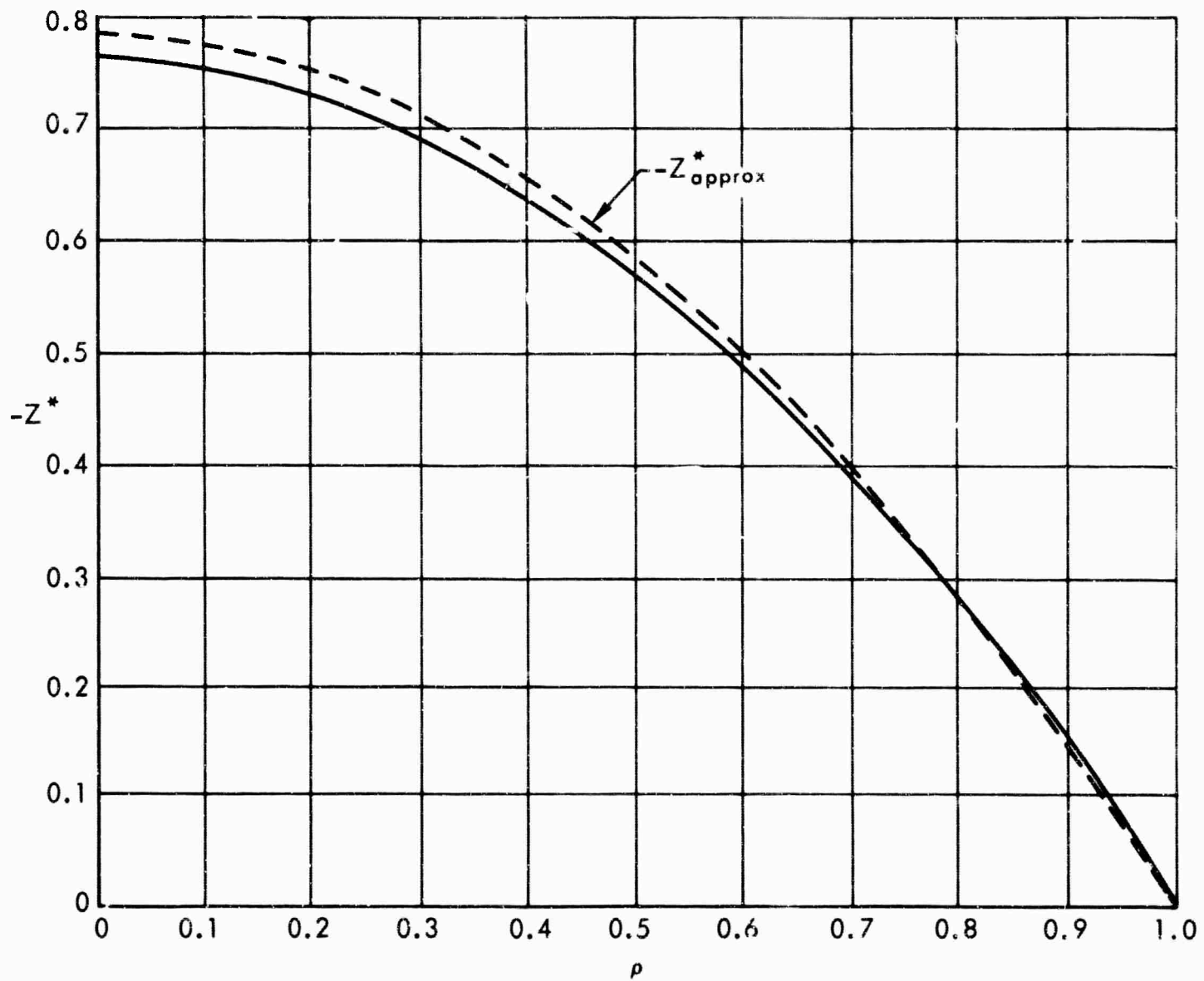
Fig.3—Curve on which w and $\dfrac{dw}{d\zeta}$ vanish

$$
\left.
\begin{aligned}
e^{\overset{\star}{\phi}_{MRZ}} &\approx \frac{e^{\overset{\star}{\phi}_B}}{\left(1 + \frac{|Z|\rho_B}{\sqrt{2}} e^{\phi_B/2}\right)^2} \qquad \frac{|Z|\rho_B}{\sqrt{2}} e^{\phi_B/2} < 1 \\[2em]
e^{\phi_{MRZ}} &\approx e^{-u} \csc^2 2\zeta = e^{-u} \csc^2\left(\frac{2Z}{1-\rho^2}\right) \\[2em]
&\qquad \frac{\sqrt{2}}{\rho_B} e^{-\phi_B/2} < |Z| \le |Z^*(\rho)| \\[2em]
e^{\overset{\star}{\phi}_{MRZ}} &= e^{-u} \qquad |Z| \ge |Z^*(\rho)|
\end{aligned}
\right\}
\tag{35}
$$

Figure 4 plots Eq. (34) along the axis $\rho = 0$ for $\rho_B = 100$ and $\phi_B = -1$.

Solution Outside the Hole for Large ρ_B

For $Z > 0$, $w = \phi$. If $-\phi_B$ is not larger than one, then $|w|$ will be less than one for the entire region $Z > 0$.

For $Z < 0$, $w = \phi + u$. We have already determined that $u < 1$ outside the hole. If $-\phi_B \sim 1$, $|w|$ will also be small.

Therefore, we can linearize Eq. (1) outside the hole for both positive and negative Z:

$$
\frac{1}{\rho_B^2}\left[\frac{\partial^2 w}{\partial Z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right)\right] - w = 0 \qquad Z > 0 \tag{36a}
$$

$$
\frac{1}{\rho_B^2}\left[\frac{\partial^2 w}{\partial Z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right)\right] - e^{-u} w = 0 \qquad \begin{array}{l} Z < 0 \\ \rho \ge 1 \end{array} \tag{36b}
$$

The function e^{-u} in Eq. (36b) is equal to $1/e$ at $\rho = 1$ and becomes equal to one within about a Debye length from the edge of the cylinder. The solutions to Eq. (36) that satisfy the boundary conditions that $|w|$ vanishes as ρ and $|Z|$ become infinite decrease exponentially with
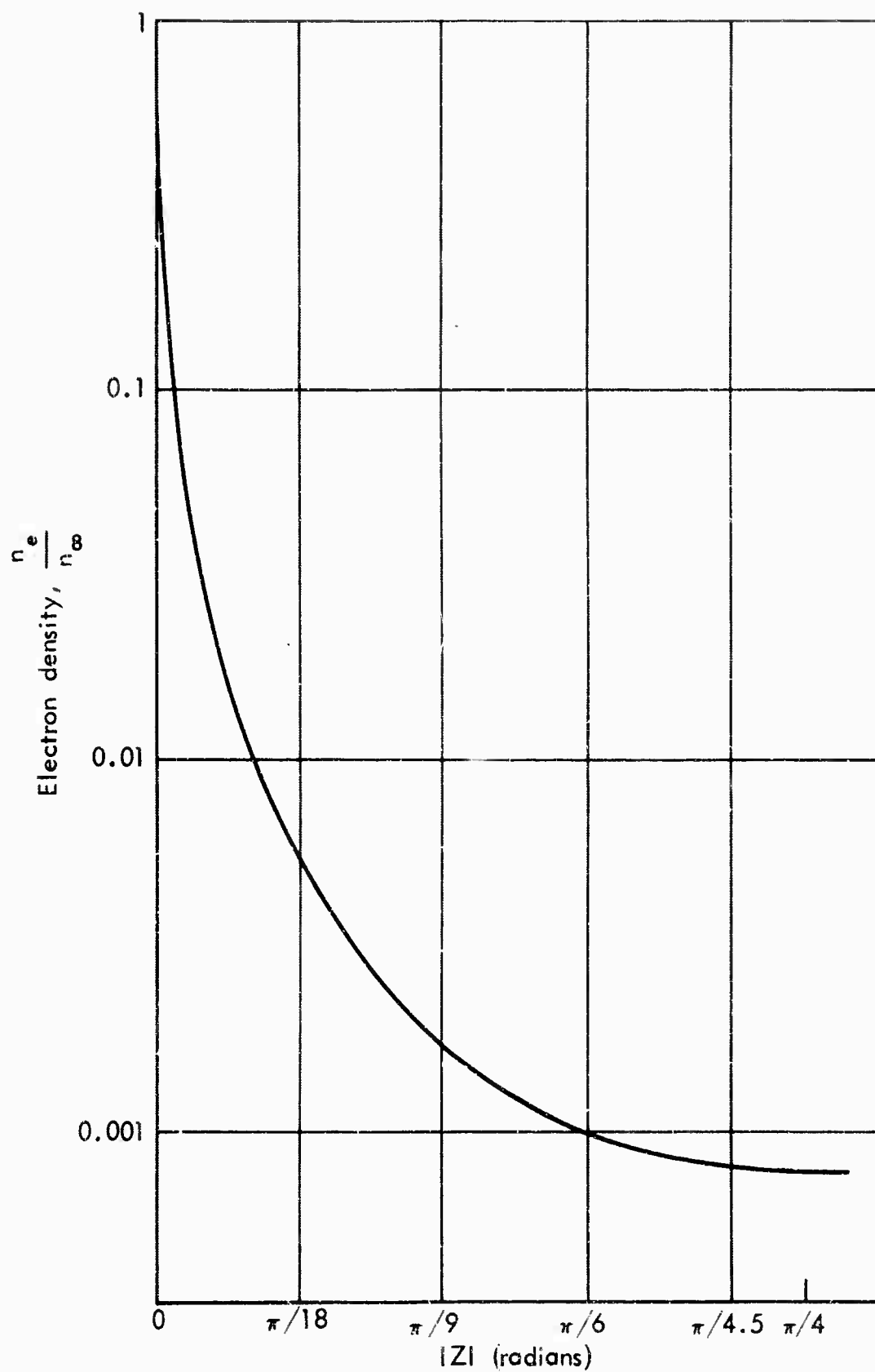
Fig. 4—Electron density behind the body at $\rho = 0$ for $\rho_B = 100$ Debye lengths ($\Phi_B = -1$)

the coordinates measured in Debye lengths from the boundaries of the semi-infinite cylindrical hole. In light of the approximations involved, finding exact solutions seems pointless. It suffices to say that within one or two Debye lengths from the boundaries of the cylindrical hole, the electron density is equal to its ambient value.

IV. BOUNDARY $Z_G(\rho)$ BETWEEN THE GUREVICH ZONES

Between the maximum-rarefaction zone, in which the zero-ion-temperature solution for the electron density should be a good approximation to the actual solution, and the region in which equating the electron and ion densities provides a good approximation to the actual solution, there exists a transition region in which neither approximation is valid. Within this transition region, the terms e^{ϕ}, n_I/n_∞, and $(1/\rho_B^2)\nabla^2\phi$ are all of the same order of magnitude, and we cannot justifiably neglect any one of the three terms.

The surface $Z = Z_G(\rho)$, upon which the zero-ion-temperature solution for e^{ϕ} is equal to the ion density (divided by n_∞), will lie in the heart of this transition region. Due to the imprecise nature of this "boundary," we will find $Z_G(\rho)$ using the simplified approximation to n_I/n_∞ (which we shall call n_ν/n_∞) obtained by treating the ions as neutral particles. The electric field will not alter the order of magnitude of the ion density; including its effect on n_I for the purpose of obtaining $Z_G(\rho)$ would simply shift the surface $Z = Z_G(\rho)$ slightly within the transition region. The magnetic field at the distances we are considering here ($|Z_G| \ll R_L/R_B$) is irrelevant.

If the scattering in the electric and magnetic fields is neglected, the ion density, based on a Maxwellian distribution at infinity, is given by

$$\frac{n_I}{n_\infty} \approx \frac{n_\nu}{n_\infty} = \frac{2V_B}{V_I\sqrt{\pi}} \int_0^\infty \exp\left[-\frac{\rho^2 u^2}{\tilde{Z}^2} - \frac{V_B^2}{V_I^2}(\tilde{Z} - u)^2\right] du \int_{u/\tilde{Z}}^\infty xe^{-x^2} I_0\left(\frac{2\rho u x}{\tilde{Z}}\right) dx$$

(37)

where

$$\tilde{Z} = -Z\frac{V_I}{V_B} \qquad Z < 0$$

and where I_0 is the hyperbolic Bessel function finite at the origin.
For $\rho = 0$, Eq. (37) becomes

$$\frac{n_v}{n_o}(\rho = 0) = \left\{ \exp - \left[\frac{1}{\tilde{z}^2 + V_I^2/V_B^2} \right] \right\} \frac{\tilde{z}}{\sqrt{\tilde{z}^2 + V_I^2/V_B^2}} \frac{1}{\sqrt{\pi}} \int_{y_1}^{\infty} e^{-y^2} \, dy \qquad (38)$$

where

$$-y_1 = \frac{V_B}{V_I} \frac{\tilde{z}}{\sqrt{\tilde{z}^2 + V_I^2/V_B^2}}$$

$$\frac{n_v}{n_\infty}(\rho = 0) \approx \frac{\tilde{z}}{\sqrt{\tilde{z}^2 + V_I^2/V_B^2}} \exp - \left[\left(\frac{1}{\tilde{z}^2 + V_I^2/V_B^2} \right) \right] \qquad \tilde{z} > V_I^2/V_B^2$$

The boundary $Z_G(\rho)$ between the Gurevich zones, as we define it, is found by equating n_I/n_∞ as given by Eq. (37) to e^{-u} as given by Eq. (19). In Fig. 5, $\tilde{Z}_G(\rho)$ is plotted for $\rho_B = 100$ and $V_I^2/V_B^2 = 0.04$, 0.01, and 0. The latter case corresponds to letting $V_I/V_B \to 0$ in Eq. (37) while keeping $\tilde{Z}$ finite. If we do this, Eq. (37) reduces to the much simpler expression

$$\frac{n_v}{n_\infty} = 2e^{-\rho^2/\tilde{z}^2} \int_{1/\tilde{z}}^{\infty} xe^{-x^2} I_o\left(\frac{2\rho}{\tilde{z}} x\right) dx \qquad (39)$$

We can see from Fig. 5 that for the parameters we have chosen, the three curves are close together. In general, we can establish a criterion for replacing Eq. (37) by Eq. (39) for the purpose of finding $Z_G(\rho)$, by looking at the expression for $\tilde{Z}_G(0)$:

$$\frac{\tilde{Z}_G(0)}{\sqrt{\tilde{Z}_G^2(0) + V_I^2/V_B^2}} \exp - \left(\frac{1}{\tilde{Z}_G^2(0) + V_I^2/V_B^2} \right) = e^{-u_o} = \frac{8}{\rho_B^2} \qquad (40)$$

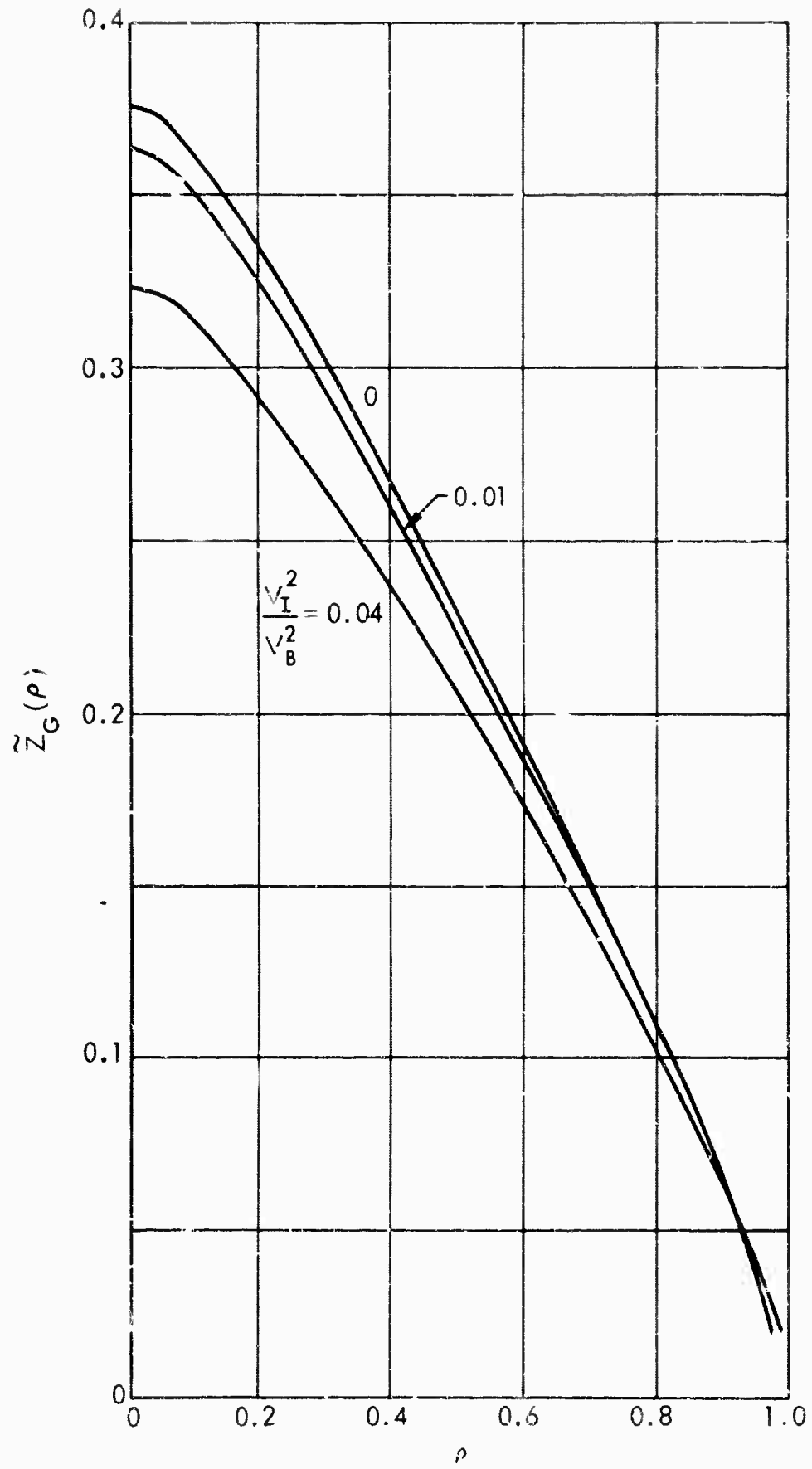If V_B^2/V_I^2 is large compared to $\ell n \, (\rho_B^2/8)$, then

Fig.5—Boundary between the two Gurevich zones

$$\tilde{Z}_G(0) \approx \left[\left(1 - \frac{V_I^2}{2V_B^2} \ell n \frac{\rho_B^2}{8}\right)\right] \frac{1}{\sqrt{\ell n \ (\rho_B^2/8)}} \cdot \tag{41}$$

will not depend much upon V_I/V_B.

We recall that the maximum-rarefaction zone is divided into two subzones: $|Z| \leq |Z^*(\rho)|$ and $|Z| \geq |Z^*(\rho)|$. Therefore, our lowest approximation to Φ for $Z < 0$ is given by

$$\Phi_I = -u(\rho) + w(\rho, z) \qquad |Z| \leq |Z^*(\rho)| \qquad\qquad \rho < 1$$

$$\tag{42}$$

$$\Phi_{II} = -u(\rho) \qquad\qquad |Z^*(\rho)| \leq |Z| \leq \frac{V_B}{V_I} \tilde{Z}_G(\rho) \qquad \rho < 1$$

$$\Phi_{III} = \ell n \frac{n_I}{n_\infty} \qquad\qquad \text{all other negative Z}$$

The location of the three regions, I, II, and III, is shown in Fig. 6 for $\rho_B = 100$ and $V_B/V_I = 10$. Figure 7 shows few equipotentials based on our zero-order approximations to Φ for $\rho_B = 100$, $V_B/V_I = 10$, $\Phi_B = -1$, and $|Z| < V_B/V_I$.
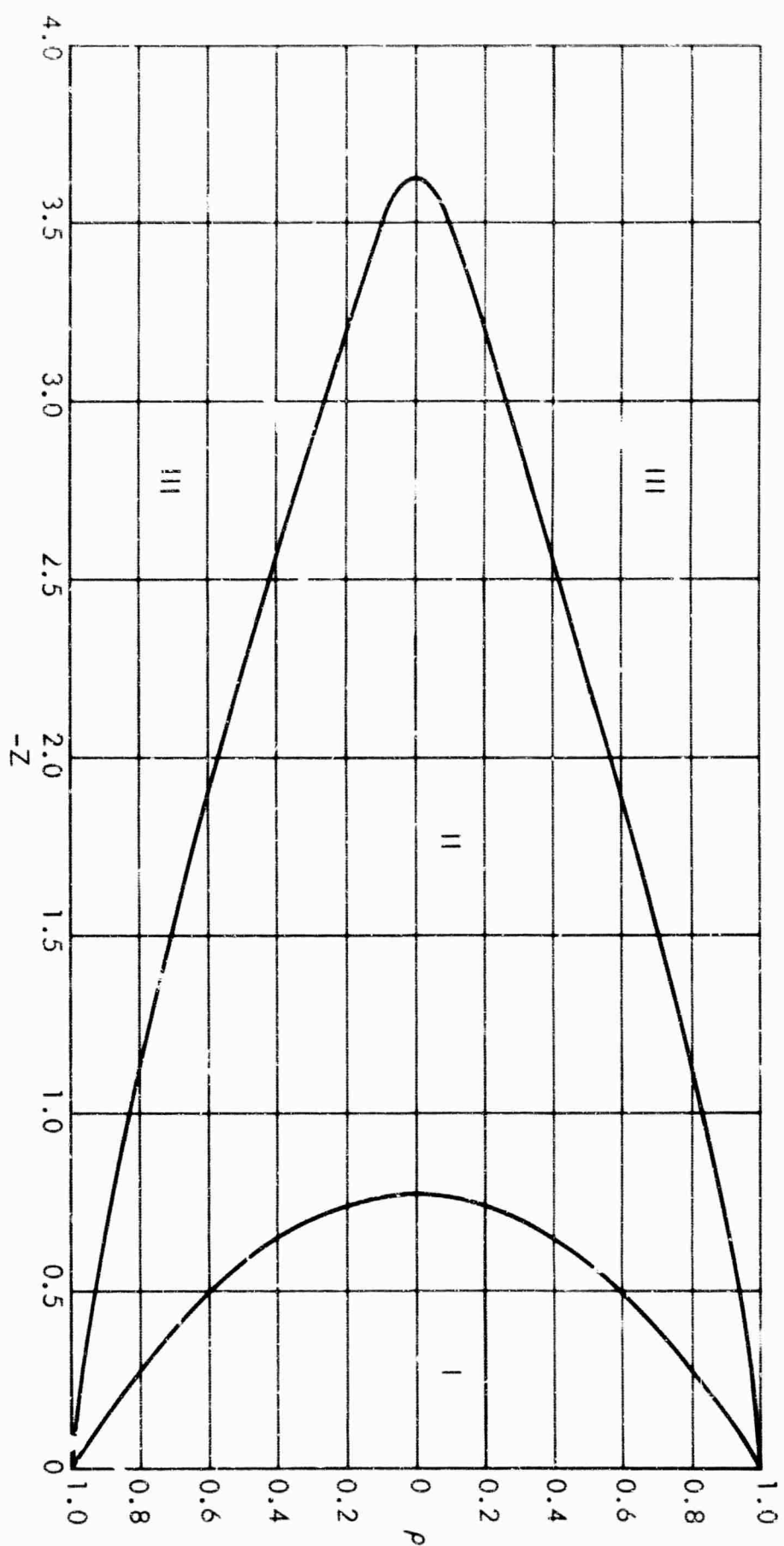
Fig. 6—The three regions behind the body for $\rho_B = 100$ and $\frac{V_B}{V_1} = 10$
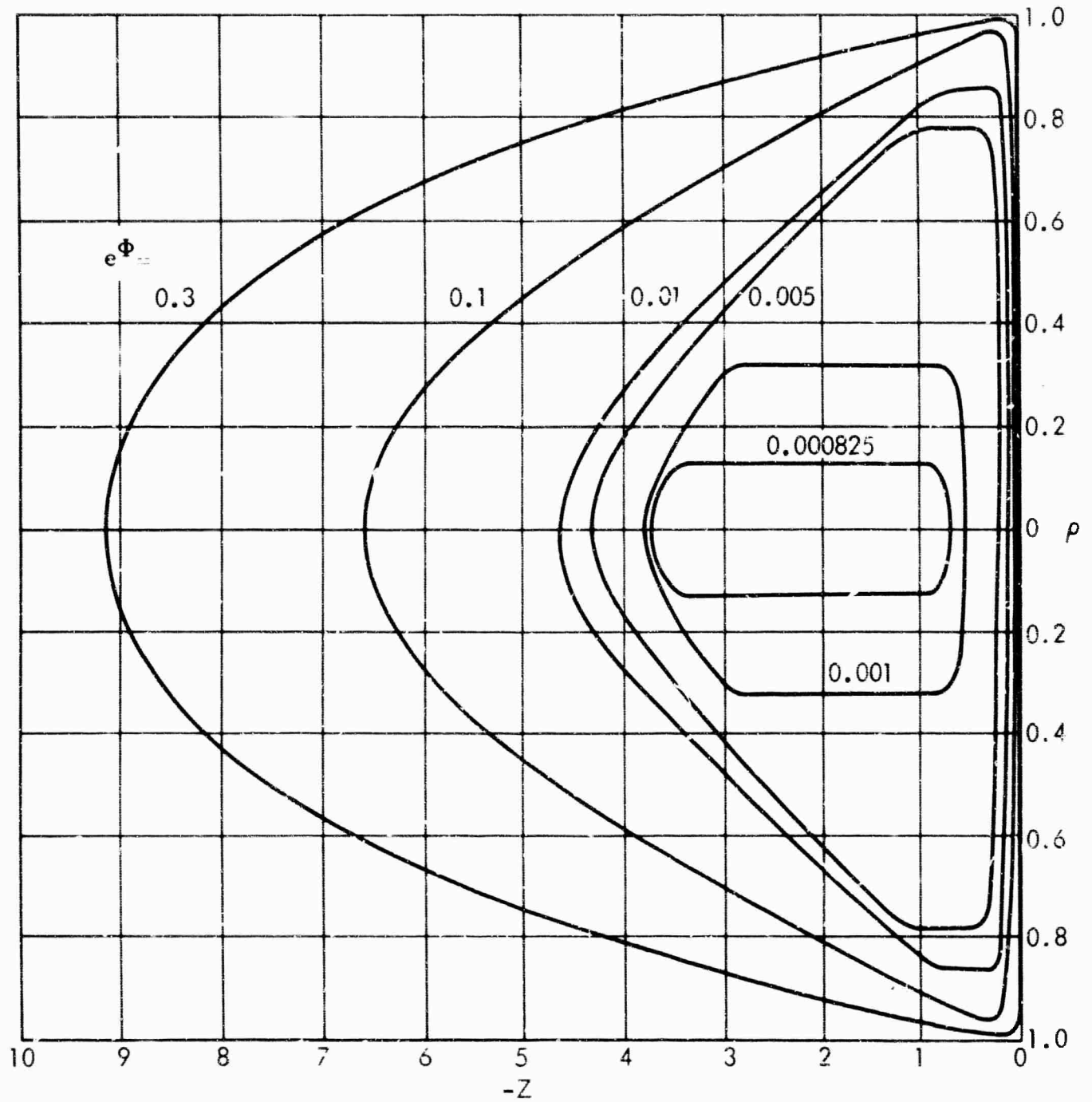
Fig. 7 —Equipotentials for $\rho_B = 100$, $\dfrac{V_B}{V_I} = 10$, and $\Phi_B = -1$

V. ION DENSITY FOR $\tilde{Z} > 1$

In order to determine the ion density, let us look at the collision-free steady-state Boltzmann equation for the ion distribution function $f_I(\vec{R}, \vec{V})$:

$$(\vec{V} - \vec{V}_B) \cdot \vec{\nabla} f_I - \frac{T_e}{T_I} \frac{V_I^2}{2} \vec{\nabla}\phi \cdot \vec{\nabla}_V f_I$$

$$+ \frac{V_B}{\rho_L} (\vec{V} \times \vec{e}_o) \cdot \vec{\nabla}_V f_I = S \tag{43}$$

where $\vec{e}_o = \vec{B}/B$ is the unit vector in the direction of the magnetic field $\vec{B}$, and $\rho_L = (V_B/eB/m_I c)1/R_B$ is the dimensionless ion Larmor radius based on the speed V_B. The right-hand side of Eq. (43) (represented by S) is a source term representing the change in the distribution function due to the uptake of ions by the disk; S can be written as

$$S = \delta(Z)(V_Z - V_B) \frac{[1 + \text{sign}(1 - \rho)]}{2} \frac{[1 + \text{sign}(V_B - V_Z)]}{2} f_I \tag{44}$$

We will use the usual technique of Fourier transforms[2] to obtain a solution of Eq. (43) in the region where small-perturbation theory is applicable ($|n_I - n_\infty|/n_\infty < 1$). The result for the Fourier transform n_K of $(n_I - n_\infty)/n_\infty$, based on a Maxwellian distribution at infinity, is given by[2]

$$n_K = - \frac{T_e}{T_I} \phi_K (1 + iN) - 2\pi \frac{J_1(\varkappa)}{\varkappa} \frac{N}{K_Z} \tag{45}$$

where

ϕ_K = the Fourier transform of ϕ

K_Z = the Z-component of $\vec{K}$

$\varkappa = \sqrt{K_x^2 + K_y^2}$

and where

$$N = \int_0^\infty d\eta \ \exp -\left[\frac{\mu_{||}^2 \eta^2}{2} + \lambda_{\perp}^2 \left(1 - \cos \frac{\eta}{\rho_L K_Z}\right) - i\eta\right]$$

$$\mu_{||}^2 = \frac{K_{||}^2 \ V_I^2}{2K_Z^2 V_B^2}$$

$$\lambda_{\perp}^2 = \frac{K_{\perp}^2 \rho_L^2}{2} \frac{V_I^2}{V_B^2}$$

$$K_{||}^2 = (\vec{K} \cdot \vec{e}_o)^2$$

$$K_{\perp}^2 = K^2 - K_{||}^2$$

We can relate ϕ_K to n_K by means of the Fourier transform or Poisson's equation. However, we are here concerned with distances that will allow us to equate the electron and ion densities. Therefore, we have

$$\phi_K \approx n_K = -2\pi \ \frac{J_1(\varkappa)}{\varkappa K_Z} \ \frac{N}{1 + \frac{T_e}{T_I} (1 + iN)} \tag{46}$$

Before considering the inversion of the Fourier transform, it would be useful to consider n_K itself under various circumstances, since

for some problems we are more interested in n_k than in $n - n_\infty$. (For example, in the region where small-perturbation theory is applicable, the scattering cross section of radiation from the wake is proportional to $|n_K|^2$, where the vector $\vec{K}$ refers to the difference in the propagation vectors of the scattered wave and the incident wave.)

FOURIER TRANSFORM OF PERTURBED PARTICLE DENSITY

We will investigate the integral N in various limits.

$\rho_L \rightarrow \infty$

In the limit of infinite Larmor radius (negligible magnetic field), we can expand the cosine in the exponent of the integrand:

$$N \rightarrow \int_0^\infty d\eta \, \exp - \left(\frac{\mu^2 \eta^2}{2} - i\eta \right) = \sqrt{\frac{\pi}{2}} \frac{1}{\mu} e^{-\frac{1}{2\mu^2}}$$

$$+ i \frac{\sqrt{2}}{\mu} \int_0^{\frac{1}{\sqrt{2}\mu}} \exp\left(x^2 - \frac{1}{2\mu^2} \right) dx$$

$$\mu^2 = \frac{K^2}{2K_Z^2} \frac{V_\perp^2}{V_B^2} \tag{47}$$

The second term can be approximated for μ small and large:

$$\frac{\sqrt{2}}{\mu} \int_0^{\frac{1}{\sqrt{2}\mu}} \exp\left(x^2 - \frac{1}{2\mu^2} \right) dx \approx 1 + \mu^2 \qquad \mu^2 \ll 1$$

$$\tag{48}$$

$$\approx \frac{1}{\mu^2} \qquad \mu^2 \gg 1$$

Thus, when the magnetic field is negligible, n_K is given by

$$n_K \approx -2\pi \, \frac{J_1(\varkappa)}{\varkappa} \, \frac{i}{K_Z} \qquad\qquad K_Z^2 \gg \frac{V_I^2}{V_B^2} \, \varkappa^2 \qquad (49a)$$

$$n_K \approx -2\pi \, \frac{J_1(\varkappa)}{\varkappa} \, \frac{\sqrt{\pi}\,V_B}{KV_I} \, \frac{1}{1 + T_e/T_I} \qquad K_Z^2 \ll \frac{V_I^2}{V_B^2} \, \varkappa^2 \qquad (49b)$$

If the electric field had been neglected, Eq. (49a) would be un-affected. The effect of the electric field for $K_Z^2 < V_I^2/V_B^2 \, \varkappa^2$ (Eq. (49b)) is to reduce $|n_K|$ by a factor $1/(1 + T_e/T_I)$.

$\underline{\lambda^2 \ll 1; \; \vec{K} \cdot \vec{e}_o \neq 0}$

For small values of λ^2, the integral N is approximately

$$N \approx \int_0^\infty d\eta \, \exp -\left(\frac{\mu_{||}^2}{2} \, \eta^2 - i\eta \right) \qquad (50)$$

so that

$$n_K \approx -2\pi \, \frac{J_1(\varkappa)}{\varkappa} \, \frac{i}{K_Z} \qquad\qquad K_Z^2 \gg \frac{V_I^2}{V_B^2} \, (\vec{K} \cdot \vec{e}_o)^2 \qquad (51a)$$

$$n_K \approx -2\pi \, \frac{J_1(\varkappa)}{\varkappa} \, \sqrt{\pi} \, \frac{V_B}{V_I} \, \frac{1}{\vec{K} \cdot \vec{e}_o} \, \frac{1}{1 + T_e/T_I} \qquad K_Z^2 \ll \frac{V_I^2}{V_B^2} \, (\vec{K} \cdot \vec{e}_o)^2 \qquad (51b)$$

For the special case of $\lambda_\perp$ identically zero $(K^2 = K_{||}^2)$, we have

$$\mu_{||}^2 = \frac{V_I^2}{2(\vec{e}_o \cdot \vec{V}_B)^2} \qquad (52)$$

so that

$$n_K \approx -2\pi i \frac{J_1(\varkappa)}{\varkappa K} \frac{\vec{e}_o \cdot \vec{V}_B}{V_B} \qquad\qquad \frac{V_I^2}{(\vec{e}_o \cdot \vec{V}_B)^2} \ll 1 \qquad (53a)$$

$$n_K = -2\pi \frac{J_1(\varkappa)}{\varkappa K} \sqrt{\pi} \frac{V_B}{V_I} \frac{1}{1 + T_e/T_I} \qquad\qquad \frac{V_I^2}{(\vec{e}_o \cdot \vec{V}_B)^2} \gg 1 \qquad (53b)$$

$\underline{\vec{K} \cdot \vec{e}_o \rightarrow 0; \ \vec{e}_o \times \vec{V}_B \neq 0}$

If $\vec{K} \cdot \vec{e}_o \rightarrow 0$, and $\vec{V}_B$ is not parallel to $\vec{B}$, then $\mu_{||} = 0$. For this case, we write

$$e^{\left[\lambda_\perp^2 \cos \frac{\eta}{\rho_L K_z}\right]} = \sum_{\ell=-\infty}^{\infty} I_\ell(\lambda_\perp^2) \ e^{i \frac{\eta}{\rho_L K_z} \ell} \qquad (54)$$

so that

$$N = \sum_{\ell=-\infty}^{\infty} e^{-\lambda_\perp^2} I_\ell(\lambda_\perp^2) \int_o^\infty e^{i\eta\left[1 + \frac{\ell}{\rho_L K_z}\right]} d\eta \qquad (55)$$

Whenever $\rho_L K_z$ is an integer, N becomes infinite. However, the integral N appears in both the numerator and denominator of Eq. (46), so that n_K itself is finite and is given by

$$n_K = 2\pi i \frac{T_I}{T_e} \frac{J_1(\varkappa)}{\varkappa \ell} \rho_L \qquad\qquad K_z \rho_L = \ell \qquad (56)$$

where ℓ is an integer.

For $\rho_L K_Z$ not equal to an integer, and $\lambda_\perp^2$ small, n_K is given by Eq. (51a).

$\vec{V}_B$ Parallel to $\vec{B}$; $\lambda^2 < 1$

If the body is moving parallel to the field, $\mu_\parallel$ is independent of K and is equal to

$$\mu_\parallel^2 = (V_I^2/2V_B^2) \ll 1 \tag{57}$$

Equation (51a) will be applicable, with $\varkappa < 1/\rho_L$, $V_I/V_B \ll 1$, so that

$$n_K \approx \frac{-2\pi i}{K_Z} \frac{J_1(\varkappa)}{\varkappa} \approx \frac{-\pi i}{K_Z} \tag{58}$$

For motion parallel to the magnetic field, n_K will go to infinity as $1/K_Z$ as $K_Z \to 0$.

SPACE DEPENDENCE OF PERTURBED ION (OR ELECTRON) DENSITY FOR $\tilde{Z} > 1$

The major difficulty in inverting the Fourier transform is provided by the factor $1/[1 + (1 + iN)T_e/T_I]$, which is a consequence of the electric field. However, we have seen that for large $\tilde{Z}$ (small $K_Z(V_B/V_I)$), the integral N is small in absolute value, and the effect of the electric field is to reduce the magnitude of the perturbation by a factor of approximately $1/(1 + T_e/T_I)$ (except for motion parallel to the field, which we will treat separately).

Therefore, we will approximate n_K for large values of $\tilde{Z}$ by

$$n_K \approx -2\pi \frac{J_1(\varkappa)}{\varkappa K_Z} \frac{N}{1 + T_e/T_I} \tag{59}$$

The inversion for $\rho \ll \rho_L$ and $\rho \ll \tilde{Z}$ can be performed in a straightforward manner. The result is given by

$$\frac{n_I - n_\infty}{n_\infty} \approx$$

$$- \frac{1}{(1 + T_e/T_I)(2\rho_L V_I^2/V_B^2)\left|\sin Z/2\rho_L\right|\sqrt{Z^2 \sin^2 \psi + 4\rho_L^2 \cos^2 \psi \sin^2 (Z/2\rho_L)}}$$

$$(60)$$

where

$$\frac{|Z|}{2\rho_L} \neq \ell\pi$$

and

$$\cos \psi = \vec{e}_o \cdot \frac{\vec{V}_B}{V_B} \neq 1$$

For $|Z|/2\rho_L < 1$, Eq. (60) becomes

$$\frac{n_I - n_\infty}{n_\infty} \approx - \frac{1}{1 + T_e/T_I}\frac{1}{\tilde{Z}^2} = \frac{1}{1 + T_e/T_I}\frac{n_\nu - n_\infty}{n_\infty} \qquad (61)$$

As expected, for distances smaller than the ion Larmor radius, the magnetic field does not affect the solution. Thus, for $(V_B/V_I)R_B < |z| < R_L$, where $|z|$ is the actual distance behind the body, we have an inverse-square potential.

For $|Z|/2\rho_L > \cot \psi$, we have

$$\frac{n_I - n_\infty}{n_\infty} \approx - \frac{1}{(1 + T_e/T_I)\left|\sin Z/2\rho_L\right|(2\rho_L V_I/V_B)\tilde{Z} \sin \psi} \qquad (62)$$

For distances large compared to R_L, the potential oscillates about a mean which decays as $1/|Z|$.

For $\psi \approx 0$, Eq. (59) does not apply for $|Z| > \rho_L$. For motion parallel to the field, the parameter μ^2 is independent of $\vec{K}$ and is small for all K_Z. Therefore, we have seen $(1 + iN) \to 0$, and we have

$$u_K \approx -2\pi \frac{J_1(\varkappa)}{\varkappa K_Z} N \tag{63}$$

$$\frac{n_I - n_\infty}{n_\infty} \cong - \frac{1}{(4\rho_L^2 \, V_I^2/V_B^2) \, \sin^2 (Z/2\rho_L)} \qquad \frac{|Z|}{2\rho_L} \neq \ell\pi; \; |Z| > \rho_L \tag{64}$$

The oscillations in Φ for $\psi = 0$ and for $|Z| > \rho_L$ are about a nonzero mean which is independent of Z, implying an extension of the perturbation to infinity. In practice, the magnetic field is never strictly uniform, and maintaining a value of ψ identically zero is impossible. Moreover, the effect of collisions (which we have completely neglected) would be to restore the medium to its unperturbed state at some distance behind the body.

VI.   EFFECT OF BODY SHAPE AND MATERIAL

Throughout this Memorandum it has been assumed that the body is a conducting disk. It is clear, however, that for distances behind the body larger than the body radius, we would obtain essentially the same solutions for any body with a circular cross section in the plane normal to $\vec{V}_B$.

Moreover, the solutions behind the body for $|Z| \gtrsim 1$ do not depend upon the potential on the body (for $|\phi_B|$ of order unity) or on the surface interactions of the charged particles with the body. The solutions in this Memorandum for $|Z| \gtrsim 1$ then can be applied to any body of circular cross section.

If the shape of the body in the plane normal to $\vec{V}_B$ is other than circular, we can modify our solutions for $\widetilde{Z} > 1$.

All the Fourier transforms (Eqs. (45) to (58)) can be modified by noting that the effect of body shape on the solution behind the body is contained in the source term. If S_K is the Fourier transform of $S/(V_Z - V_B)f_\infty$, we have

$$S_K = \iint e^{-i\vec{K}\cdot\vec{R}'} \, d\vec{R}' \tag{65}$$

where the double integral is over the body surface in the plane normal to $\vec{V}_B$. For our circular cross section,

$$S_K = 2\pi \frac{J_1(\varkappa)}{\varkappa} \tag{66}$$

For other shapes, we can modify all the Fourier transforms (Eqs. (45) to (58)) by replacing $2\pi(J_1(\varkappa)/\varkappa)$ by Eq. (65).

For the space dependence of n_I at large distances for arbitrary body shapes, we modify Eqs. (60), (61), (62), and (64) by multiplying the expression for $(n_I - n_\infty)/n_\infty$ by the factor $A/\pi K_B^2$, where A is the area of the body surface normal to $\vec{V}_B$. Thus, for example, Eq. (60) is given by

$$\frac{n_I - n_\infty}{n_\infty} \approx$$

$$\frac{-A}{\tau(1 + T_e/T_I)(2R_L \, V_I^2/V_B^2) \, \left|\sin(z/2R_L)\right| \sqrt{z^2 \sin^2 \psi + 4R_L^2 \cos^2 \psi \, \sin^2(z/2R_L)}}$$

$$(6.)$$

The solution within the maximum-rarefaction zone will depend upon the shape of the body. For any given geometry, the order of magnitude of $e^{\psi_{MRZ}}$ can be determined from Poisson's equation.

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An investigation of the steady-state perturbations of ion and electron densities caused by a large-radius conducting disk moving through the ionosphere at a speed intermediate to the ion and electron thermal speeds. An approximation of the electron density in the region close behind the body is found by solving Poisson's equation, neglecting the ion density. For larger distances behind the body, where ion and electron densities are almost equal, an expression for the Fourier transform of the perturbed ion density is obtained, taking account of the ion thermal motion and scattering in the electric and magnetic fields, but neglecting collisions.	Ionosphere Wake Radar